HOSSAM GHANEM

(19) 8.2 Trigonometric integrals (A)

FORMULAS

$\cos^2 x + \sin^2 x = 1$	$\sin 2x = 2\sin x \cos x$
$1 + \tan^2 x = \sec^2 x$	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
$1 + \cot^2 x = \csc^2 x$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\int \sin x \; dx = -\cos x + C$	$\int \tan x \; dx = -\ln\lvert\cos x\rvert + C$
$\int \cos x \; dx = \sin x + C$	$\int \cot x \; dx = \ln\lvert\sin x\rvert + C$
$\int \sec^2 x \; dx = \tan x + C$	$\int \sec x \; dx = \ln\lvert\sec x + \tan x\rvert + C$
$\int \csc^2 x \; dx = -\cot x + C$	$\int \csc x \; dx = \ln\lvert\csc x - \cot x\rvert + C$
$\int \sec x \tan x \; dx = \sec x + C$	$\int e^u \, du = e^u + C$
$\int \csc x \cot x \; dx = -\csc x + C$	$\int \frac{1}{u} \, du = \ln\lvert u\rvert + C$

Trigonometric integrals

أولا تكاملات تحتوي على $\cos x$, $\sin x$

الحالة الأولى

تحتوي على $\cos x$, $\sin x$ مرفوعين لأسس على الصورة $\cos^n x$, $\sin^m x$ بحيث **أحد الأسين فردي**
نسحب واحد من النسبة ذات الأس الفردي ونحول الباقي للنسبة الأخرى

مثال :

1	هنا الـ $\text{Sin}^3 x$ ذات أس فردي (3)	$\int \text{Sin}^3 x \ \cos^2 x \, dx$
2	تم سحب $\sin x$ من $\text{Sin}^3 x$ فتصبح $\sin^2 x$	$\int \sin^2 x \cdot \sin x \cdot \cos^2 x \, dx$
3	تم تحويل $\sin^2 x$ إلي $(1-\cos^2 x)$	$\int (1-\cos^2 x)\cos^2 x \cdot \sin x \, dx$
4	استخدم التعويض	$t = \cos x$
5	ثم اشتق لتحصل على الـ dt	$dt = -\sin t$ $-dt = \sin x \, dx$
6	عوض في التكامل الناتج من الخطوة رقم 3	$-\int (1-t^2)t^2 \cdot dt$

ملحوظات

-1 إذا كان أسس الـ $\cos x$, $\sin x$ كلاهما فردي نسحب من الأس الأقل

2 – $\sin^4 x = (\sin^2 x)^2 = (1-\cos^2 x)^2$

-3 هذه الطريقة تسري على التكاملات مثل $\int \cos^3 x \ dx$, $\int \cos^4 x \sin x \ dx$ أيضا

الحالة الثانية

تحتوي على $\cos x$, $\sin x$ مرفوعين لأسس على الصورة $\cos^n x$, $\sin^m x$ بحيث **الأسين كلهما زوجي**
يتم حل التكامل بالاختصارات مستعين بالقوانين الآتية

$\sin^2 x + \cos^2 x = 1$	1
$\cos^2 x = \frac{1}{2}(1+\cos 2x)$	2
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$	3
$\frac{\sin x}{\cos^2 x} = \sec x \tan x$	4
$\frac{\cos x}{\sin^2 x} = \csc x \cot x$	5
$\sin 2x = 2\sin x \cos x$	6
$\cos 2x = \cos^2 x - \sin^2 x$ $\cos 2x = 2\cos^2 x - 1$ $\cos 2x = 1 - 2\sin^2 x$	7

Example 1 Evaluate the following integral $\int \tan^2 x \ \cos^5 x \, dx$ 18 Dec. 1999

Solution

$$I = \int \tan^2 x \ \cos^5 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \cos^5 x \, dx$$
$$= \int \sin^2 x \ \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cdot \cos x \, dx$$
$$= \int \sin^2 x \,(1 - \sin^2 x) \cos x \, dx$$

Let $u = \sin x$ $\qquad du = \cos x \ dx$

$$I = \int u^2 (1 - u^2) \, du = \int (u^2 - u^4) \ du$$
$$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + c_1$$
$$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + c$$

Example 2 Evaluate the following integrals. (3 pts) $\int \tan^5 x \ \ dx$ 55 July 23 , 2011

Solution

$$I = \int \tan^5 x \ \ dx = \int \frac{\sin^5 x}{\cos^5 x} \ dx = \int \sin^5 x \cos^{-5} x \ dx = \int \sin^4 x \cos^{-5} x \sin x \ dx$$
$$= \int \sin^4 x \cos^{-5} x \sin x \ dx = \int (\sin^2 x)^2 \cos^{-5} x \sin x \ dx = \int (1 - \cos^2 x)^2 \cos^{-5} x \sin x \ dx$$

Let $u = \cos x$ $\qquad du = -\sin x \ dx$

$$I = -\int (1 - u^2)^2 u^{-5} du = -\int (1 - 2u^2 + u^4) u^{-5} du = \int (-u^{-5} + 2u^{-3} - u^{-1}) du$$
$$= \frac{1}{4} u^{-4} - u^{-2} - \ln|u| + c = \frac{1}{4} (\cos x)^{-4} - (\cos x)^{-2} - \ln|\cos x| + c$$

Example 3 Evaluate the following integral $\int \frac{\sin^5 x}{\sqrt{\cos x}} dx$ 13 May 1998

Solution

$$I = \int \frac{\sin^5 x}{\sqrt{\cos x}} dx = \int \frac{\sin^4 x}{\sqrt{\cos x}} \cdot \sin x \ dx$$
$$= \int \frac{(\sin^2 x)^2}{(\cos x)^{\frac{1}{2}}} \cdot \sin x \ dx = \int \frac{(1 - \cos^2 x)^2}{(\cos x)^{\frac{1}{2}}} \sin x \ dx$$

Let $u = \cos x$ $\qquad du = -\sin x \ dx$

$$I = -\int \frac{(1 - u^2)^2}{(u)^{\frac{1}{2}}} \ du = -\int \frac{(1 - 2u^2 + u^4)}{(u)^{\frac{1}{2}}} \ du$$
$$= -\int \left(u^{\frac{-1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{7}{2}} \right) du$$
$$= -\left(2u^{\frac{1}{2}} - 2 \cdot \frac{2}{5} u^{\frac{5}{2}} + \frac{2}{9} u^{\frac{9}{2}} \right) + c$$
$$= \frac{4}{5} \cos^{\frac{5}{2}} x - \ 2 \cos^{\frac{1}{2}} x - \frac{2}{9} \cos^{\frac{9}{2}} x + c$$

Example 4 Evaluate the following ($2\,{}^{1}/_{2}$ points) $\int_0^{\pi/8} \frac{1-\tan^2 x}{\sec^2 x}\,dx$ 56 11 December 2011

Solution

$$I=\int_0^{\frac{\pi}{8}} \frac{1-\tan^2 x}{\sec^2 x}\,dx=\int_0^{\frac{\pi}{8}} \frac{\cos^2 x}{\cos^2 x}\cdot\frac{1-\tan^2 x}{\sec^2 x}\,dx=\int_0^{\frac{\pi}{8}}(\cos^2 x-\sin^2 x)dx=\int_0^{\frac{\pi}{8}}\cos 2x\,dx$$

$$=\frac{1}{2}\Big[\sin 2x\Big]_0^{\frac{\pi}{8}}=\frac{1}{2}\left(\sin\frac{\pi}{4}-\sin 0\right)=\frac{1}{2}\cdot\frac{1}{\sqrt{2}}=\frac{1}{2\sqrt{2}}=\frac{1}{4}\sqrt{2}$$

Example 5 Evaluate the following integral $\int \tan^2 x\sin^2 x\,dx$

Solution

$$I=\int \tan^2 x\sin^2 x\,dx=\int \frac{\sin^2 x}{\cos^2 x}\cdot\sin^2 x\;\;dx=\int\frac{\sin^4 x}{\cos^2 x}\;dx$$

$$=\int\frac{(1-\cos^2 x)^2}{\cos^2 x}\;dx=\int\frac{1-2\cos^2 x+\cos^4 x}{\cos^2 x}dx\int\left(\frac{1}{\cos^2 x}+\frac{-2\cos^2 x}{\cos^2 x}+\frac{\cos^4 x}{\cos^2 x}\right)dx$$

$$=\int(\sec^2 x-2+\cos^2 x)\;\;dx=\tan x-2x+\frac{1}{2}\int 1+\cos 2x\;\;dx$$

$$=\tan x-2x+\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)=\tan x-2x+\frac{1}{2}x+\frac{1}{4}\sin 2x+c=\tan x+\frac{1}{4}\sin 2x-\frac{3}{2}x+c$$

Example 6 Evaluate the following integral $\int\frac{\cos^2 x}{\sin^4 x}\,dx$

Solution

$$I=\int\frac{\cos^2 x}{\sin^4 x}\;dx=\int\frac{\cos^2 x}{\sin^2 x}\cdot\frac{1}{\sin^2 x}\;dx=\int\cot^2 x\cdot\csc^2 x\;\;dx$$

Let $u=\cot x$ $du=-\csc^2 x\;\;dx$

$$I=-\int u^2\;\;du=-\frac{1}{3}\,u^3+c_1=-\frac{1}{3}\cot^3 x+c$$

Example 7 Evaluate the following integral $\int(\sec x)(\sin 2x)^3\,dx$

Solution

$$I=\int(\sec x)(\sin 2x)^3 dx=\int\frac{1}{\cos x}(2\sin x\cos x)^3\cdot dx=\int\frac{1}{\cos x}\cdot 8\sin^3 x\cos^3 x\,dx$$

$$=8\int\cos^2 x\sin^3 x\;\;dx=8\int\cos^2 x\sin^2 x\cdot\sin x\;\;dx=8\int\cos^2 x\,(1-\cos^2 x)\cdot\sin x\;\;dx$$

Let $u=\cos x$ $du=-\sin x\;\;dx$

$$I=-8\int u^2(1-u^2)\;\;du=-8\int(u^2-u^4)\,du=-\frac{8}{3}\,u^3+\frac{8}{5}\,u^5+c_1$$

$$=\frac{8}{5}\cos^5 x-\frac{8}{5}\cos^3 x+c$$

Example 8 Evaluate the following integral $\int \sin^2 x \sin(4x)\, dx$ 7 Nov. 1996

Solution

$$I = \int \sin^2 x \sin(4x)\, dx = \int \frac{1}{2}(1-\cos 2x)\cdot 2\sin 2x \cos 2x\, dx = \frac{1}{2}\int (1-\cos 2x)\cos 2x \cdot 2\sin 2x\ dx$$

Let $u = \cos 2x$ $du = -2\sin 2x\ dx$

$$I = -\frac{1}{2}\int (1-u)\cdot u\, du = -\frac{1}{2}\int (u^2 - u)\, du$$

$$= -\frac{1}{2}\left(\frac{1}{3}u^3 - \frac{1}{2}u^2\right) + c = -\frac{1}{6}u^3 + \frac{1}{4}u^2 + c$$

$$= -\frac{1}{6}\cos^3 2x + \frac{1}{4}\cos^2 2x + c$$

Example 9 Evaluate the following integral $\int \cot^2 x \sec x\ dx$

Solution

$$I = \int \cot^2 x \sec x\ dx = \int \frac{\cos^2 x}{\sin^2 x}\cdot\frac{1}{\cos x}\, dx = \int \frac{\cos x}{\sin^2 x}\, dx$$

$$= \int \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}\, dx = \int \cot x\ \csc x\ dx = -\csc x + c$$

Another solution

$$I = \int \cot^2 x \sec x\ dx = \int \frac{\cos^2 x}{\sin^2 x}\cdot\frac{1}{\cos x}\, dx = \int \frac{\cos x}{\sin^2 x}\, dx = \int \frac{1}{\sin^2 x}\cdot \cos x\ dx$$

Let $u = \sin x$ $du = \cos x\ dx$

$$I = \int \frac{1}{u^2}\, du = \int u^{-2}\ du = -u^{-1} + c = \frac{-1}{u} + c = \frac{-1}{\sin x} + c = -\csc x + c$$

Homework

1	Evaluate the integral	$\int \sin^5 x \cos^2 x \, dx$	32 December 2003
2	Evaluate the integral	$\int \frac{\sin^5 x}{\sec^2 x} dx$	14 Nov. 1998
3	Evaluate the integral	$\int \cot^2 x \sin^3 x \, dx$	5 May 1996
4	Evaluate the integral	$\int \frac{\sin x}{\cos^3 x} \, dx$	
5	Evaluate the integral	$\int \frac{\cos x}{\sin^3 x} \, dx$	
6	Evaluate the integral	$\int \sin^3 x \sec x \, dx$	
7	Evaluate the integral	$\int \sqrt{\sec x} \sin^3 x \, dx$	12 December 1997
8	Evaluate the integral	$\int \sin^3 x \sqrt{\sec x} \, dx$	22 December 2000
9	Evaluate the integral	$\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$	13 May 1998
10	Evaluate the integral	$\int \frac{\cos^3 x}{\sqrt{\sin x}} dx$	16 May 1999
11	Evaluate the integral	$\int \cos^3 x \sqrt{\csc x} \, dx$	34 July 2004
12	Evaluate the integral	$\int (\sec x)^{\frac{4}{3}} \sin x \, dx$	4 December 1995
13	Evaluate the integral	$\int \cos^3 x \sqrt{\sin^3 x} \, dx$	38 July 2005
14	Evaluate the integral	$\int \frac{\sin^3 x}{1 - \sqrt{\cos x}} dx$	24 August 2001
15	Evaluate the integral	$\int \frac{\cot^3 x}{\sin^{\frac{3}{2}} x} dx$	23 May 2001

Homework

16	Evaluate the integral	$\int \frac{\sin x}{\sqrt{1+\cos x}}\,dx$	
17	Evaluate the integral	$\int \frac{\cot x}{\csc x - 1}\,dx$	4 December 1995
18	Evaluate the integral	$\int \frac{\cos^4 x}{\sin^2 x}\,dx$	
19	Evaluate the integral	$\int \frac{\sin^2 x}{\cos^4 x}\,dx$	
20	Evaluate the integral	$\int \frac{\sin x}{\sin 2x}\,dx$	
21	Evaluate the integral	$\int \frac{\cos 2x}{\cos x}\,dx$	
22	Evaluate the integral	$\int \csc^3 2x\ \sin^6 x\,dx$	41 July 2006 A
23	Evaluate the integral	$\int \frac{\sin^2(2x)}{\cos(2x)}\,dx$	
24	Evaluate the following	$\int \frac{\sin(2x)}{\sec^3 x}\,dx$ ($2\,{}^1/_2$ points)	56 11 December 2011
25	Evaluate the integral	$\int \csc x \tan x\ dx$	
26	Evaluate the following integral [3 marks e]	$\int \sin^3 x \sec^{1/3} x\ dx$	52 July 24, 2010
27	(4 pts.) Evaluate	$\int_0^{\pi/2} \sin x \cos x \cos(4x)\,dx.$	53 11 December 2010
28	Evaluate the following integral	$\int \sec^3 x \sin x\ dx$	
29	Evaluate the following integral	$\int \frac{\csc^3 x\ \cot^3 x}{\sqrt{\sin x}}\,dx$	

28 Evaluate the following integral $\int \sec^3 x \sin x \; dx$

Solution

$$I = \int \sec^3 x \sin x \; dx = \int \frac{1}{\cos^3 x} \sin x \; dx$$

Let $u = \cos x \qquad \to du = -\sin x \; dx$

$$I = -\int \frac{1}{u^3} \, du = -\int u^{-3} \, du = -\frac{-1}{2} u^{-2} + c = \frac{1}{2} (\cos x)^{-2} + c_1 = \frac{1}{2} \sec^2 x + c$$

Another solution

$$I = \int \sec^3 x \sin x \; dx = \int \sec^2 x \sec x \sin x \; dx = \int \sec^2 x \frac{\sin x}{\cos x} \, dx = \int \tan x \; \sec^2 x \; dx$$

$u = \tan x \qquad \to du = \sec^2 x \; dx$

$$I = \int u \, du = \frac{1}{2} u^2 + c = \frac{1}{2} \tan^2 u + c$$

29 Evaluate the following integral $\int \frac{\csc^3 x \; \cot^3 x}{\sqrt{\sin x}} dx$

Solution

$$I = \int \frac{\csc^3 x \; \cot^3 x}{\sqrt{\sin x}} dx = \int \frac{1}{\sin^3 x} \cdot \frac{1}{\sin^{\frac{1}{2}} x} \cdot \frac{\cos^3 x}{\sin^3 x} \; dx$$

$$= \int \sin^{\frac{-13}{2}} x \cdot \cos^2 x \cos x \; dx$$

$$= \int \sin^{\frac{-13}{2}} x \; (1 - \sin^2 x) \cos x \; dx$$

$u = \sin x \qquad \to du = \cos x \; dx$

$$I = \int u^{\frac{-13}{2}} (1 - u^2) \, du = \int \left(u^{\frac{-13}{2}} - u^{\frac{-9}{2}} \right) du$$

$$= -\frac{2}{11} u^{\frac{-11}{2}} + \frac{2}{7} u^{\frac{-7}{2}} + c$$

$$= -\frac{2}{11} \sin^{\frac{-11}{2}} x + \frac{2}{7} \sin^{\frac{-7}{2}} x + c$$